

Second order iterative functional equations related to a competition equation

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Dedicated to Professor János Aczél on the occasion of his 90th birthday

Abstract. The functional equation related to competition ([2])

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad x, y \in \mathbb{R}, xy \neq 1,$$

for $y = cx$ with a fixed $c > 0$, leads to the equation

$$f\left(\frac{(1+c)x}{1-cx^2}\right) = \frac{f(x)+f(cx)}{1+f(x)f(cx)}, \quad x \in \mathbb{R}, |x| < \frac{1}{\sqrt{c}}.$$

The case $c = 1$ (a first order iterative functional equation) was treated in [3]. In this paper we consider the case $c \neq 1$ (when the equation is of the second order). We show that a function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$, differentiable at the point 0 satisfies this functional equation iff there is a real p such that $f = \tanh \circ (p \tan^{-1})$ which extends the main result of [3].

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1. Introduction

The functional equation on a *restricted domain* (following [1] a *conditional* functional equation),

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad x, y \in \mathbb{R}, xy \neq 1, \quad (\text{CE})$$

the so-called *competition equation*, considered first in [2], was also treated in [3]. If f is a solution then either $f(0) = 0$, or $f(0) = -1$ or $f(0) = 1$. It is

shown in [2] (cf. also [3]) that, if f takes value -1 or 1 , then it is a constant function (cf. also Remark 1). Therefore only the case $f(0) = 0$ is interesting. The main result of [3] says that, in this case, a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the equation for all real x, y such that $xy < 1$, iff $f = \tanh \circ \alpha \circ \tan^{-1}$ where $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is an additive function. Thus, if f is measurable or continuous at least at one point, then there is a $p \in \mathbb{R}$ such that $f = g_p$ where

$$g_p = \tanh \circ (p \tan^{-1}).$$

Taking $y = x$ in (CE) we obtain

$$f\left(\frac{2x}{1-x^2}\right) = \frac{2f(x)}{1+f(x)^2}, \quad x \in \mathbb{R}, x^2 \neq 1,$$

the diagonalization of the competition equation, which is a first order iterative functional equation (cf. [5, 6]). In [2] it is proved that if a function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that $f(0) = 0$, satisfies this equation for all $x \in (-1, 1)$, and is twice differentiable at the point 0, then $f = g_p$ for some real p .

In [4] the following stronger result is presented. *A function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$, differentiable at the point 0, satisfies this functional equation iff there is a real p such that $f = g_p$. Moreover $g'_p(0) = p$ (Theorem 1).* Applying the theory of iterative functional equations ([5, 6], cf. also [7, 8]) it is also shown that in this result the assumption of the differentiability of the solution f at the point 0 cannot be replaced by the continuity of f at 0.

In the present paper we consider a generalization of the above diagonalization problem. Namely, in Sect. 2, for a fixed $c > 0$, restricting the competition equation to the straight line $y = cx$, we obtain the iterative functional equation

$$f\left(\frac{(1+c)x}{1-cx^2}\right) = \frac{f(x) + f(cx)}{1 + f(x)f(cx)}, \quad x \in \mathbb{R}, |x| < \frac{1}{\sqrt{c}},$$

which is of the second order if $c \neq 1$. (For the definition of the n th order of an iterative functional equation see [5], chapter XII, and [6] pp. 237–239.) Our main result says that a function $f : \mathbb{R} \rightarrow \mathbb{R}$, differentiable at the point 0 and such that $f(0) = 0$, satisfies this equation iff there is a $p \in \mathbb{R}$ such that $f = g_p$, where $g_p = \tanh \circ (p \tan^{-1})$.

In Sect. 3 we discuss the case when $c \leq 0$.

In Sect. 4, recalling the motivation coming from a meteorological phenomenon (hail suppression by competition of small particles via cloud seeding), we discuss the mutual relation between (CE) and a Riccati differential equation.

2. Main result

Let us fix $c > 0$. In this section we consider the competition equation restricted to the straight line $\{(x, cx) : x \in \mathbb{R}\}$.

Theorem 1. Let $c > 0$ be arbitrarily fixed. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the point 0 and $f(0) = 0$. If f satisfies the functional equation

$$f\left(\frac{(1+c)x}{1-cx^2}\right) = \frac{f(x) + f(cx)}{1 + f(x)f(cx)}, \quad |x| < \frac{1}{\sqrt{c}}, \quad (1)$$

then there is a $p \in \mathbb{R}$ such that $f = g_p$, where $g_p = \tanh \circ (p \tan^{-1})$.

Proof. It is easy to verify that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the point 0, $f(0) = 0$, and f satisfies equation (1), then the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(x) := \begin{cases} \frac{f(x)}{x} & \text{for } x \neq 0 \\ p & \text{for } x = 0 \end{cases},$$

where

$$p := f'(0),$$

is continuous at 0 and satisfies the functional equation

$$\frac{(1+c)}{1-cx^2} \varphi\left(\frac{(1+c)x}{1-cx^2}\right) = \frac{\varphi(x) + c\varphi(cx)}{1 + cx^2\varphi(x)\varphi(cx)}, \quad |x| < \frac{1}{\sqrt{c}}. \quad (2)$$

Suppose that the functions $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$ satisfy this equation, are continuous at the point 0 and

$$\varphi_1(0) = \varphi_2(0) = p.$$

Put

$$I_c := \left(-\frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}\right).$$

The function $h : I_c \rightarrow \mathbb{R}$ defined by

$$h(x) = \frac{(1+c)x}{1-cx^2}, \quad x \in I_c,$$

is odd, continuous, strictly increasing, convex in $(0, \infty)$, concave in $(-\infty, 0)$, maps I_c onto \mathbb{R} and

$$h'(0) = 1 + c.$$

It follows that its inverse $\beta := h^{-1}$ is odd, continuous, increasing, convex in $(-\infty, 0)$, concave in $(0, \infty)$, maps \mathbb{R} onto I_c and

$$\beta'(0) = \frac{1}{1+c} < 1.$$

Of course, the function $\gamma := c\beta$ has similar properties. Moreover

$$\gamma'(0) = \frac{c}{1+c} < 1,$$

$$0 < \frac{\beta(t)}{t} < 1, \quad 0 < \frac{\gamma(t)}{t} < 1, \quad t \in \mathbb{R}, t \neq 0,$$

and, consequently, for every $t \in \mathbb{R}$, the sequences $(\beta^n)_{n \in \mathbb{N}}$ and $(\gamma^n)_{n \in \mathbb{N}}$ of iterates of the functions β and γ converge uniformly to 0 on compact subsets of \mathbb{R} .

With the above solutions φ_1, φ_2 of equation (2), we have

$$\frac{(1+c)}{1-cx^2} \varphi_j \left(\frac{(1+c)x}{1-cx^2} \right) = \frac{\varphi_j(x) + c\varphi_j(cx)}{1+cx^2\varphi_j(x)\varphi_j(cx)}, \quad x \in I_c, \quad j = 1, 2.$$

Hence, putting

$$\psi = \varphi_1 - \varphi_2,$$

and subtracting the respective sides of these equalities for $j = 1$ and $j = 2$, we get, for all $x \in I_c$,

$$\begin{aligned} \varphi_1 \left(\frac{(1+c)x}{1-cx^2} \right) - \varphi_2 \left(\frac{(1+c)x}{1-cx^2} \right) &= \psi \left(\frac{(1+c)x}{1-cx^2} \right) \\ &= \frac{1-cx^2}{1+c} \frac{\psi(x) [1-c^2x^2\varphi_1(cx)\varphi_2(cx)] + c\psi(cx) [1-x^2\varphi_1(x)\varphi_2(x)]}{[1+cx^2\varphi_1(x)\varphi_1(cx)][1+cx^2\varphi_2(x)\varphi_2(cx)]}. \end{aligned}$$

Since φ_1 and φ_2 are continuous at 0 and $\varphi_1(0) = \varphi_2(0)$, there is $\delta > 0$ such that

$$\begin{aligned} 1-c^2x^2\varphi_1(cx)\varphi_2(cx) &> 0; \quad 1-x^2\varphi_1(x)\varphi_2(x) > 0; \\ 1+cx^2\varphi_j(x)\varphi_j(cx) &> 0 \quad (j = 1, 2), \end{aligned}$$

and $1-cx^2 > 0$ for all $x \in (-\delta, \delta)$. Hence, setting

$$\begin{aligned} k(x) &:= \frac{1-cx^2}{1+c} \frac{1-c^2x^2\varphi_1(cx)\varphi_2(cx)}{[1+cx^2\varphi_1(x)\varphi_1(cx)][1+cx^2\varphi_2(x)\varphi_2(cx)]}, \\ m(x) &:= \frac{1-cx^2}{1+c} \frac{c[1-x^2\varphi_1(x)\varphi_2(x)]}{[1+cx^2\varphi_1(x)\varphi_1(cx)][1+cx^2\varphi_2(x)\varphi_2(cx)]}, \end{aligned}$$

we have

$$k(x) > 0, \quad m(x) > 0, \quad x \in (-\delta, \delta),$$

and

$$\left| \psi \left(\frac{(1+c)x}{1-cx^2} \right) \right| \leq k(x) |\psi(x)| + m(x) |\psi(cx)|, \quad x \in (-\delta, \delta). \quad (3)$$

Moreover, for $x \in (-\delta, \delta)$, we have

$$\begin{aligned} k(x) + m(x) &\leq \frac{1}{1+c} \frac{1-c^2x^2\varphi_1(cx)\varphi_2(cx) + c[1-x^2\varphi_1(x)\varphi_2(x)]}{[1+cx^2\varphi_1(x)\varphi_1(cx)][1+cx^2\varphi_2(x)\varphi_2(cx)]} \\ &\leq \frac{1}{1+c} \frac{1+c}{[1+cx^2\varphi_1(x)\varphi_1(cx)][1+cx^2\varphi_2(x)\varphi_2(cx)]} \leq 1. \end{aligned}$$

Setting $x = \beta(t)$ in inequality (3) and considering the definition of γ , we get

$$|\psi(t)| \leq \kappa(t) |\psi(\beta(t))| + \mu(t) |\psi(\gamma(t))|, \quad t \in I := \beta^{-1}(-\delta, \delta), \quad (4)$$

where $\kappa(t) = k(\beta(t))$, $\mu(t) = m(\beta(t))$, hence

$$\kappa(t) + \mu(t) \leq 1, \quad t \in I, \quad (5)$$

and, of course, $I = \beta^{-1}(-\delta, \delta)$ is an open interval such that $0 \in I$.

Define $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ by

$$\sigma(t) := \begin{cases} \max(\beta(t), \gamma(t)) & \text{for } t \geq 0 \\ \min(\beta(t), \gamma(t)) & \text{for } t < 0 \end{cases}.$$

Of course, σ is continuous, strictly increasing and

$$0 < \frac{\sigma(t)}{t} < 1, \quad t \in \mathbb{R}, t \neq 0.$$

From inequalities (4) and (5) we obtain, for all $t \in I$,

$$\begin{aligned} |\psi(t)| &\leq \kappa(t) \sup |\psi(\beta(I))| + \mu(t) \sup |\psi(\gamma(I))| \\ &\leq (\kappa(t) + \mu(t)) \sup |\psi(\sigma(I))| = \sup |\psi(\sigma(I))|, \end{aligned}$$

whence

$$\sup |\psi(I)| \leq \sup |\psi(\sigma(I))|.$$

Hence, by induction,

$$\sup |\psi(I)| \leq \sup |\psi(\sigma^n(I))|, \quad n \in \mathbb{N},$$

where σ^n denotes the n -th iterate of σ . Since the decreasing sequence of intervals $(\sigma^n(I))_{n \in \mathbb{N}}$ tends to $\{0\}$, this inequality, the continuity of ψ at 0 and $\psi(0) = 0$ imply that $\sup |\psi(I)| = 0$. Hence, from the definition of ψ we get $\varphi_1(x) = \varphi_2(x)$ for all $x \in I$. Now, according to the theory of iterative functional equations (cf. [5], p. 68, Lemma 3.1) we conclude that $\varphi_1 = \varphi_2$. Taking into account the definition of the function φ , we have proved that, for any $p \in \mathbb{R}$, there exists at most one solution f of equation (1) that is differentiable at 0 and such that $f'(0) = p$. Since, in view of Lemma 1, the function $f = g_p$ satisfies equation (1), is differentiable at 0 and $f'(0) = p$, the proof is complete. \square

To justify the assumption $f(0) = 0$ consider the following

Remark 1. Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution of the competition equation (CE). Setting $x = y = 0$ in (CE) we obtain $f(0) = \frac{2f(0)}{1+[f(0)]^2}$ whence either $f(0) = 0$ or $f(0) = 1$ or $f(0) = -1$.

If $f(0) = 1$ then, setting $y = 0$ in (CE), we get $f(x) = 1$ for all $x \in \mathbb{R}$.

If $f(0) = -1$ then, setting $y = 0$ in (CE), we get $f(x) = -1$ for all $x \in \mathbb{R}$.

3. Discussion of the case when c is nonpositive

In this section we consider the functions $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$, satisfying the competition equation (CE) restricted to the straight line $\{(x, cx) : x \in \mathbb{R}\}$, i.e. the equation

$$f\left(\frac{(1+c)x}{1-cx^2}\right) = \frac{f(x) + f(cx)}{1 + f(x)f(cx)},$$

where $c \leq 0$.

Remark 2. For $c = 0$ this equation reduces to the identity $f(x) = f(x)$ for all $x \in \mathbb{R}$, so every function $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(0) = 0$, is a solution.

For $c = -1$ this equation reduces to the equality $0 = f(x) + f(-x)$, so $f : \mathbb{R} \rightarrow \mathbb{R}$ is a solution iff it is an odd function.

Remark 3. The counterpart of Theorem 1 holds true in the case $c < 0$ and $c \neq -1$.

Assume, for instance that $-1 < c < 0$; setting $y = -cx$ in the competition equation (CE) we get

$$f\left(\frac{(1-c)x}{1+cx^2}\right) = \frac{f(x) + f(-cx)}{1 + f(x)f(-cx)}, \quad x \in \mathbb{R},$$

whence, calculating $f(x)$, we get the functional equation

$$f(x) = \frac{f\left(\frac{(1-c)x}{1+cx^2}\right) - f(-cx)}{1 - f(-cx)f\left(\frac{(1-c)x}{1+cx^2}\right)}, \quad x \in \mathbb{R}.$$

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the point 0, $f(0) = 0$, and f satisfies this equation, then the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\varphi(x) := \begin{cases} \frac{f(x)}{x} & \text{for } x \neq 0 \\ p & \text{for } x = 0 \end{cases},$$

where

$$p := f'(0),$$

is continuous at 0 and satisfies the functional equation

$$\varphi(x) = \frac{\frac{1-c}{1+cx^2}\varphi\left(\frac{(1-c)x}{1+cx^2}\right) + c\varphi(-cx)}{1 + \frac{c(1-c)x^2}{1+cx^2}\varphi(-cx)\varphi\left(\frac{(1-c)x}{1+cx^2}\right)}, \quad x \in \mathbb{R},$$

whence, setting

$$h(x) := \frac{(1-c)x}{1+cx^2}, \quad x \in \mathbb{R},$$

we can write this equation in the form

$$\varphi(x) = \frac{(1-c)\varphi(h(x)) + c(1+cx^2)\varphi(-cx)}{1+cx^2 + c(1-c)x^2\varphi(-cx)\varphi(h(x))}, \quad x \in \mathbb{R}.$$

Suppose that the functions $\varphi_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$ satisfy this equation, are continuous at the point 0 and

$$\varphi_1(0) = \varphi_2(0) = p.$$

Hence we have

$$\varphi_j(x) = \frac{(1-c)\varphi_j(h(x)) + c(1+cx^2)\varphi_j(-cx)}{1+cx^2 + c(1-c)x^2\varphi_j(-cx)\varphi_j(h(x))}, \quad x \in \mathbb{R}; j = 1, 2.$$

Setting

$$\begin{aligned} \psi &:= \varphi_1 - \varphi_2, & y_1 &= y_1(x) = \varphi_1(h(x)), & y_2 &= y_2(x) = \varphi_1(-cx), \\ z_1 &= z_1(x) = \varphi_2(h(x)), & z_2 &= z_2(x) = \varphi_2(-cx), & (x \in \mathbb{R}), \end{aligned}$$

and subtracting the respective sides of these equalities for $j = 1$ and $j = 2$, performing simple calculations, we get

$$\begin{aligned} \psi(x) &= \frac{(1-c)[1+cx^2 - c^2x^2(1+cx^2)y_2z_2]\psi(h(x)) + [c(1+cx^2) - c(1-c)^2x^2y_1z_1]\psi(-cx)}{[1+cx^2 + c(1-c)x^2y_1y_2][1+cx^2 + c(1-c)x^2z_1z_2]} \end{aligned}$$

for $x \in \mathbb{R}$, whence

$$\psi(x) = \lambda(x)\psi(h(x)) + \mu(x)\psi(-cx), \quad x \in \mathbb{R}, \quad (6)$$

where

$$\begin{aligned} \lambda(x) &:= \frac{(1-c)[1+cx^2 - c^2x^2(1+cx^2)y_2z_2]}{[1+cx^2 + c(1-c)x^2y_1y_2][1+cx^2 + c(1-c)x^2z_1z_2]}, & x \in \mathbb{R}, \\ \mu(x) &:= \frac{c(1+cx^2) - c(1-c)^2x^2y_1z_1}{[1+cx^2 + c(1-c)x^2y_1y_2][1+cx^2 + c(1-c)x^2z_1z_2]}, & x \in \mathbb{R}. \end{aligned}$$

Since the functions y_1, y_2, z_1, z_2 are continuous at $x = 0$ and

$$y_1(0) = y_2(0) = z_1(0) = z_2(0) = p,$$

it is easy to check that there is a real number $\delta > 0$ such that

$$\lambda(x) \geq 0, \quad \mu(x) \geq 0, \quad \lambda(x) + \mu(x) \leq 1, \quad x \in (-\delta, \delta).$$

Hence, making use of (6), we obtain

$$|\psi(x)| \leq \lambda(x)|\psi(h(x))| + \mu(x)|\psi(-cx)|, \quad x \in (-\delta, \delta).$$

Now we can argue similarly as in the proof of Theorem 1.

4. Motivation and remarks on a Riccati-type differential equation and the competition functional equation (CE)

In [2] it was shown that a model of a meteorological phenomenon in cloud physics, interpreted as competition and described with the aid of a Riccati differential equation, can be fully characterized by the functional equation (CE) that does not involve any derivative. The functional equation (CE) reflects symmetry properties of the corresponding model differential equation (and of its solution).

Remark 4. (Differential equation and initial condition)

Some physical models present themselves in this way,

$$\frac{dF(X)}{dX} = \frac{1 - b[F(X)]^2}{1 + aX^2}, \quad X \in \mathbb{R}, \quad (7)$$

with *two* arbitrary parameters $a, b > 0$, fixed. (In [2] there is $a = b = 1$.) This may be reduced to a *one*-parameter model: (7) can be written in the form

$$\frac{d \left[\sqrt{b} F \left(\frac{1}{\sqrt{a}} (\sqrt{a} X) \right) \right]}{\sqrt{b} \left(\frac{1}{\sqrt{a}} d(\sqrt{a} X) \right)} = \frac{1 - \left[\sqrt{b} F \left(\frac{1}{\sqrt{a}} (\sqrt{a} X) \right) \right]^2}{1 + (\sqrt{a} X)^2},$$

whence

$$\frac{d \left[\sqrt{b} F \left(\frac{1}{\sqrt{a}} (\sqrt{a} X) \right) \right]}{d(\sqrt{a} X)} = \frac{\sqrt{b}}{\sqrt{a}} \frac{1 - \left[\sqrt{b} F \left(\frac{1}{\sqrt{a}} (\sqrt{a} X) \right) \right]^2}{1 + (\sqrt{a} X)^2}.$$

Therefore, setting

$$f(x) = \sqrt{b} F \left(\frac{1}{\sqrt{a}} x \right), \quad x = \sqrt{a} X,$$

we can write this equation in the form

$$\frac{df(x)}{dx} = \frac{\sqrt{b}}{\sqrt{a}} \frac{1 - f(x)^2}{1 + x^2}.$$

Hence, if $F(0) = 0 = f(0)$ then $p = f'(0) = \frac{\sqrt{b}}{\sqrt{a}}$ and, consequently, we can write this equation in the form

$$\frac{df(x)}{dx} = p \frac{1 - f(x)^2}{1 + x^2}, \quad x \in \mathbb{R}, \quad (8)$$

a nonlinear first-order differential equation of Riccati-type. Here we have only *one* parameter, namely $p > 0$, fixed. (In [2], as $a = b = 1$, we have $f = F$ and $p = 1$.)

With initial condition $f(x_0) = f_0$ (where $x_0, f_0 \in \mathbb{R}$, fixed), the solution of equation (8) (obtainable by separation of variables) reads as

$$\begin{aligned} f(x) &= \tanh [p(\tan^{-1}(x) - \tan^{-1}(x_0)) + \tanh^{-1}(f_0)] \\ &\equiv \frac{g_p(x) - g_p(x_0) + [1 - g_p(x)g_p(x_0)] \cdot f_0}{1 - g_p(x)g_p(x_0) + [g_p(x) - g_p(x_0)] \cdot f_0} \equiv \frac{f_0 + g_p\left(\frac{x-x_0}{1+xx_0}\right)}{1 + f_0g_p\left(\frac{x-x_0}{1+xx_0}\right)}. \end{aligned} \quad (9)$$

Check: $f(x_0) = f_0$, and we have $f(0) = [f_0 + g_p(-x_0)] / [1 + f_0g_p(-x_0)] \neq 0$ (in general). It is to be noted that the last two versions of (9) do not require $|f_0| < 1$ (fixed) but admit $f_0 \in \mathbb{R}$ (fixed). In general, (9) is not an odd function of x . Only for the special initial condition $f(0) = 0$ (i.e. $x_0 = 0, f_0 = 0$, called "standard" initial condition), the result (9) reduces to an odd function (called "standard" solution),

$$f(x) = \tanh [p \tan^{-1}(x)] \equiv g_p(x). \quad (10)$$

With this standard solution g_p we can represent the general solution f of the differential equation (8) via (9) as a rational (fractional linear) expression in g_p .

From (10) we get $g_{-p}(x) = g_p(-x) = -g_p(x)$, implying that we may relax the condition $p > 0$ (fixed) to $p \in \mathbb{R}$ (fixed) for $f: \mathbb{R} \rightarrow \mathbb{R}$. Already from (8) we see that changing the sign of p is equivalent to changing the sign of f [and f_0 in (9)].

Remark 5. If (in a certain physical situation) a slightly more general *four*-parameter model appears adequate or desirable (e.g. for easy interpretation),

$$\frac{dG}{dX} = \frac{C - BG(X)^2}{D + AX^2}, \quad X \in \mathbb{R}, \quad (11)$$

with parameters $A, B, C, D \in \mathbb{R}$ (fixed) and $CD \neq 0$, it can be reduced immediately to the *two*-parameter model (7) by writing

$$\frac{D}{C} \frac{dG}{dX} = \frac{1 - \frac{B}{C}G(X)^2}{1 + \frac{A}{D}X^2} \equiv \frac{1 - \frac{BC}{D^2} \left[\frac{D}{C}G(X) \right]^2}{1 + \frac{A}{D}X^2};$$

abbreviating $a := \frac{A}{D}$, $b := \frac{BC}{D^2}$, $F(X) := \frac{D}{C}G(X)$, we get equation (7). [Further reduction to the *one*-parameter model (8) may be done as above.]

Remark 6. If a differentiable function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation (CE) then it satisfies the Riccati differential equation (8).

Proof. Assume that a differentiable function $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x) + f(y)}{1 + f(x)f(y)}, \quad x, y \in \mathbb{R}, xy < 1.$$

Differentiating both sides with respect to x we get

$$f' \left(\frac{x+y}{1-xy} \right) \frac{1+y^2}{(1-xy)^2} = \frac{f'(x) [1-f(y)^2]}{[1+f(x)f(y)]^2}, \quad x, y \in \mathbb{R}, xy < 1,$$

while differentiating with respect to y , we obtain

$$f' \left(\frac{x+y}{1-xy} \right) \frac{1+x^2}{(1-xy)^2} = \frac{f'(y) [1-f(x)^2]}{[1+f(x)f(y)]^2}, \quad x, y \in \mathbb{R}, xy < 1.$$

It follows that

$$\frac{(1-xy)^2 f'(x) [1-f(y)^2]}{1+y^2 [1+f(x)f(y)]^2} = \frac{(1-xy)^2 f'(y) [1-f(x)^2]}{1+x^2 [1+f(x)f(y)]^2}, \quad x, y \in \mathbb{R}, xy < 1,$$

whence

$$\frac{f'(x) [1-f(y)^2]}{1+y^2} = \frac{f'(y) [1-f(x)^2]}{1+x^2}, \quad x, y \in \mathbb{R}, xy < 1.$$

Thus either there is $y \in \mathbb{R}$ such that $[f(y)]^2 \neq 0$ and, consequently,

$$f'(x) = p \frac{1-[f(x)]^2}{1+x^2}, \quad x \in \mathbb{R},$$

for some fixed $p \in \mathbb{R}$, that is, f satisfies the Riccati differential equation (8), or $f = 1$ or $f = -1$. \square

The correspondence of (CE) and (8) is interesting: at first sight, the functional equation (CE) seems to lack the parameter p , whereas p shows up *explicitly* in the differential equation (8). However, p is contained implicitly in the functional equation via $p = f'(0)$; it could be called "hidden parameter". Functional equation (CE) reflects symmetry properties of differential equation (8) (and its solution) for any value of the parameter p .

The equations (CE) and (8) are not equivalent (because the families of the solutions of these equations are different). Any differentiable solution of functional equation (CE) is a solution of differential equation (8) (but not conversely). One of the advantages of a functional equation is that it sometimes permits to get some solutions of a (corresponding) differential equation without (strong) differentiability conditions. (In the present paper we assume differentiability of the solution only at one point.)

Remark 7. Let $c \in \mathbb{R}$, $c \neq 0$, be fixed. Note that a function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional equation

$$\chi \left(\frac{x+y}{1-xy} \right) = \frac{\chi(x) + \chi(y)}{1+c^2\chi(x)\chi(y)}, \quad x, y \in \mathbb{R}, xy \neq 1, \quad (12)$$

iff $f = c\chi$ satisfies functional equation (CE),

$$f\left(\frac{x+y}{1-xy}\right) = \frac{f(x)+f(y)}{1+f(x)f(y)}, \quad x, y \in \mathbb{R}, xy \neq 1.$$

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